

# $q$ -BERNOULLI NUMBERS AND POLYNOMIALS ASSOCIATED WITH GAUSSIAN BINOMIAL COEFFICIENT

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ABSTRACT. The first purpose of this paper is to present a systemic study of some families of multiple  $q$ -Bernoulli numbers and polynomials by using multivariate  $q$ -Volkenborn integral (=  $p$ -adic  $q$ -integral) on  $\mathbb{Z}_p$ . From the studies of these  $q$ -Bernoulli numbers and polynomials of higher order we derive some interesting  $q$ -analogs of Stirling number identities.

## §1. Introduction

Let  $q$  be regarded as either a complex number  $q \in \mathbb{C}$  or a  $p$ -adic number  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$ , then we always assume  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , we normally assume  $|1 - q|_p < p^{-\frac{1}{p-1}}$ , which implies that  $q^x = \exp(x \log q)$  for  $|x|_p \leq 1$ . Here,  $|\cdot|_p$  is the  $p$ -adic absolute value in  $\mathbb{C}_p$  with  $|p|_p = \frac{1}{p}$ . The  $q$ -basic natural number are defined by  $[n]_q = \frac{1-q^{n+1}}{1-q} = 1 + q + \cdots + q^n$ , ( $n \in \mathbb{N}$ ), and  $q$ -factorial are also defined as  $[n]_q! = [n]_q \cdot [n-1]_q \cdots [2]_q \cdot [1]_q$ . In this paper we use the notation of Gaussian binomial coefficient as follows:

$$(1) \quad \binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!} = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q!}.$$

Note that  $\lim_{q \rightarrow 1} \binom{n}{k}_q = \binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}$ . The Gaussian coefficient satisfies the following recursion formula:

$$(2) \quad \binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q, \text{ cf. [1-23].}$$

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From thus recursion formula we derive

$$(3) \quad \binom{n}{k}_q = \sum_{d_0 + \dots + d_k = n-k, d_i \in \mathbb{N}} q^{d_1 + 2d_2 + \dots + kd_k}, \text{ see [15, 20, 21].}$$

Let  $p$  be a fixed prime. Throughout this paper  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ ,  $\mathbb{C}$ , and  $\mathbb{C}_p$  will, respectively, denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers, the complex number field, and the completion of algebraic closure of  $\mathbb{Q}_p$ . For  $d$  a fixed positive integer  $(p, d) = 1$ , let

$$\begin{aligned} X &= X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \text{ and } X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 \leq a < dp \\ (a, p) = 1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where  $a \in \mathbb{Z}$  lies in  $0 \leq a < dp^N$ , cf. [8-18]. For  $x \in \mathbb{C}_p$ , we use the notation  $[x]_q = \frac{1-q^x}{1-q}$ , cf. [1-6].

We say that  $f$  is a uniformly differentiable function at a point  $a \in \mathbb{Z}_p$  and denote this property by  $f \in UD(\mathbb{Z}_p)$ , if the difference quotients  $F_f(x, y) = \frac{f(x) - f(y)}{x - y}$  have a limit  $l = f'(a)$  as  $(x, y) \rightarrow (a, a)$ . For  $f \in UD(\mathbb{Z}_p)$ , let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N\mathbb{Z}_p),$$

representing a  $q$ -analogue of Riemann sums for  $f$ , cf. [8, 21-23]. The integral of  $f$  on  $\mathbb{Z}_p$  will be defined as limit ( $n \rightarrow \infty$ ) of those sums, when it exists. The  $q$ -Volkenborn integral ( $=p$ -adic  $q$ -integral) of the function  $f \in UD(\mathbb{Z}_p)$  is defined by

$$(4) \quad I_q(f) = \int_X f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_q} \sum_{0 \leq x < dp^N} f(x) q^x, \text{ see [8].}$$

The Carlitz's  $q$ -Bernoulli numbers  $\beta_{k,q}$  can be determined inductively by

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing  $\beta^i$  by  $\beta_{i,q}$  (see [2, 3, 24, 25]).

In [8], it was shown that the Carlitz's  $q$ -Bernoulli numbers can be represented by  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  as follows:

$$(5) \quad \int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \int_X [x]_q^m d\mu_q(x) = \beta_{m,q}, \quad m \in \mathbb{Z}_+.$$

The  $k$ -th order factorial of the  $q$ -number  $[x]_q$ , which is defined by

$$[x]_{k,q} = [x]_q \cdot [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1}) \cdots (1-q^{x-k+1})}{(1-q)^k},$$

is called  $q$ -factorial of  $x$  of order  $k$ , cf.[15, 19, 20]. From this we note that  $\binom{x}{k}_q = \frac{[x]_{k,q}}{[k]_q!}$ , cf.[4-16]. The theory of  $q$ -number and the factorial of  $q$ -number are applicable in the many areas related to mathematics, mathematical physics and probability. For example, we consider a sequence of Bernoulli trials and assume that the conditionally probability of success at the  $n$ -th trial, given that  $k$  successes occur before that trial varies geometrically with  $n$  and  $k$ . Specifically, suppose that the probability of success at the  $n+1$ -th trial, given that  $k$  success occur up the  $n$ -th trial, is given by

$$\lambda_{n,k} = q^{an+bk+c} (\in \mathbb{R}), \quad k = 0, 1, 2, \dots, n, \quad n = 0, 1, 2, \dots,$$

with  $a, b$  and  $c$  such that  $0 \leq \lambda_{n,k} \leq 1$ . The particular case  $b = 0$  corresponds to the assumption that the probability of success at any trial depends only on the number of previous trials, while the other particular case  $a = 0$  corresponds to the assumption that the probability of success at only trial depends only on the number of previous success. The purpose of this paper is to present a systemic study of some families of multiple  $q$ -Bernoulli numbers and polynomials by using multivariate  $q$ -Volkenborn integral(= $p$ -adic  $q$ -integral) on  $\mathbb{Z}_p$ . From the studies of these  $q$ -Bernoulli numbers and polynomials we derive some interesting  $q$ -analogs of Stirling number identities. That is, the  $q$ -analogs of many classical Stirling number identities are formulated and their interesting features are revealed in this paper.

## §2. $q$ -Bernoulli numbers associated with $q$ -Stirling number identities

In this section we assume that  $q \in \mathbb{C}_p$  with  $|1-q|_p < p^{-\frac{1}{p-1}}$ . From the definition of  $[x]_q$  we can easily derive the following equation.

$$(6) \quad q^n[x-n]_q = \frac{q^n - 1 + 1 - q^x}{1-q} = [x]_q - [n]_q, \quad \text{and} \quad [-x]_q = \frac{1}{q^x} \frac{q^x - 1}{1-q} = -\frac{1}{q^x} [x]_q.$$

Let  $(Eh)(x) = h(x+1)$  be the shift operator. Then we consider the  $q$ -difference operator as follows:

$$(7) \quad \Delta_q^n = \prod_{i=1}^n (E - q^{i-1}I), \quad \text{where } (Ih)(x) = h(x).$$

From (6) and (7), we note that

$$(8) \quad f(x) = \sum_{n \geq 0} \binom{x}{n}_q \Delta_q^n f(0),$$

where

$$(9) \quad \Delta_q^n f(0) = \sum_{k=0}^n \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} f(n-k).$$

The  $q$ -Stirling number of the second kind is defined by Carlitz as follows:

$$(10) \quad s_2(n, k, q) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \sum_{j=0}^k (-1)^j q^{\binom{j}{2}} \binom{k}{j}_q [k-j]_q^n, \text{ see [3].}$$

By (9) and (10) we easily see that

$$(11) \quad s_2(n, k, q) = \frac{q^{-\binom{k}{2}}}{[k]_q!} \Delta_q^k 0^n.$$

From (11) we can also derive the following equation.

$$(12) \quad [x]^n = \sum_{k=0}^n \binom{x}{k}_q [k]_q! s_2(k, n-k, q) q^{\binom{k}{2}} = \sum_{k=0}^n [x]_{k,q} \frac{q^{\binom{k}{2}_q - \binom{n-k}{2}_q}}{[n-k]_q!} \Delta_q^{n-k} 0^k.$$

By (2), we easily see that

$$(13) \quad \int_{\mathbb{Z}_p} \binom{x}{n}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{(n+1) - \binom{n+1}{2}}.$$

From (5), (12) and (13) we can derive the following theorem.

**Theorem 1.** *For  $m \in \mathbb{Z}_+$ , we have*

$$(14) \quad \beta_{m,q} = q \sum_{k=0}^m \frac{[k]_q!}{[k+1]_q} (-1)^k s_2(k, m-k, q),$$

where  $\beta_{m,q}$  are  $m$ -th Carlitz  $q$ -Bernoulli numbers.

The  $q$ -Stirling numbers of the first kind is defined as

$$(15) \quad (1-q)^n [x]_{n,q} = \prod_{i=1}^n (1 - q^{x-n+1} q^{i-1}) = \sum_{l=0}^n \binom{n}{l}_q q^{\binom{l}{2}} (-1)^l q^{l(x-n+1)}.$$

It is easy to see that

$$(16) \quad q^{lx} = ([x]_q (q-1) + 1)^l = \sum_{m=0}^l \binom{l}{m} (q-1)^m [x]_q^m.$$

From (16) we note that

$$\begin{aligned}
(17) \quad & \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l}_q q^{\binom{l}{2}} (-1)^l q^{l(x-n+1)} \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^n \binom{n}{l}_q q^{\binom{l}{2}+l-ln} (-1)^l \sum_{m=0}^l \binom{l}{m} (q-1)^m [x]_q^m \\
&= \frac{1}{(1-q)^n} \sum_{m=0}^n (q-1)^m \left( \sum_{l=m}^n \binom{n}{l}_q q^{\binom{l}{2}-ln+l} \binom{l}{m} (-1)^l \right) [x]_q^m.
\end{aligned}$$

By (12), (15) and (17) we obtain the following theorem.

**Theorem 2.** For  $n \in \mathbb{Z}_+$  we have

$$\beta_{n,q} = \sum_{l=0}^n s_2(l, n-l, q) q^{\binom{l}{2}} \sum_{m=0}^l \frac{1}{(1-q)^{l-m}} \left( \sum_{i=m}^l \binom{l}{i}_q \binom{i}{m} q^{\binom{i}{2}-il+i} (-1)^i \right) \beta_{m,q}.$$

In [3] Carlitz has given the following relation.

$$(18) \quad s_2(n, k, q) = (q-1)^{-k} \sum_{j=0}^k (-1)^{k-j} \binom{k+n}{k-j}_q \binom{j+n}{j}_q,$$

and

$$\binom{n}{k}_q = \sum_{j=0}^n \binom{n}{j}_q (q-1)^{j-k} s_2(k, j-k, q).$$

By simple calculation we easily see that

$$(19) \quad q^{nt} = \sum_{k=0}^n (q-1)^k q^{\binom{k}{2}} \binom{n}{k}_q [t]_{k,q} = \sum_{m=0}^n \left( \sum_{k=m}^n (q-1)^k \binom{n}{k}_q s_1(k, m, q) \right) [t]_q^m.$$

By using  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  we have

$$(20) \quad \int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^n \binom{n}{m} (q-1)^m \beta_{m,q}.$$

From (19) and (20) we derive

$$(21) \quad \binom{n}{m} = \sum_{k=m}^n (q-1)^{-m+k} \binom{n}{k}_q s_1(k, m, q).$$

From the definition of the first kind Stirling number we note that

$$(22) \quad q^{\binom{n}{2}} \binom{x}{n}_q [n]_q! = [x]_{n,q} q^{\binom{n}{2}} = \sum_{k=0}^n s_1(n, k, q) [x]_q^k.$$

By (13) and (22) we have

$$(23) \quad \frac{1}{[n+1]_q} = \frac{q^{-1}}{[n]_q!} \sum_{k=0}^n (-1)^{n-k} s_1(n, k, q) \beta_{k,q}.$$

By (14), (15), (17) and (23) we obtain the following theorem.

**Theorem 3.** *For  $n, j \in \mathbb{Z}_+$  we have*

$$(24) \quad s_1(n, j, q) = \frac{q^{\binom{n}{2}}}{(q-1)^{n-j}} \sum_{k=j}^n (-1)^{n-k} q^{\binom{k+1}{2} - nk} \binom{n}{k}_q \binom{k}{j}.$$

Moreover,

$$\frac{1}{[n+1]_q} = \frac{q^{-1}}{[n]_q!} \sum_{k=0}^n (-1)^{n-k} s_1(n, k, q) \beta_{k,q}.$$

### §3. Multivariate $p$ -adic $q$ -integral on $\mathbb{Z}_p$ associated with $q$ -Stirling numbers

In this section we also assume that  $q \in \mathbb{C}_p$  with  $|1 - q|_p < p^{-\frac{1}{p-1}}$ . For any positive integers  $k, m$ , we consider the following multivariate  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  related to  $q$ -Bernoulli polynomials of higher order as follows:

$$(25) \quad \beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n (-1)^i \binom{n}{i} q^{ix} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^k (k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k).$$

In the special case  $x = 0$ ,  $\beta_{n,q}^{(k)}(0) = \beta_{n,q}^{(k)}$  will be called the  $q$ -Bernoulli numbers of order  $k$ . From (25) we note that

$$(26) \quad \beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(i+k) \cdots (i+1)}{[i+k]_q \cdots [i+1]_q} q^{ix}.$$

Thus, we obtain the following theorem.

**Theorem 4.** *For  $m, k \in \mathbb{Z}_+$ , we have*

$$\beta_{n,q}^{(k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n \frac{(-1)^i \binom{n}{i} \binom{i+k}{k}}{\binom{i+k}{k}_q} \frac{k!}{[k]_q!} q^{ix}.$$

Now we also define  $\beta_{n,q}^{(-k)}(x)$  as follows:

$$(27) \quad \beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n \frac{(-1)^i \binom{n}{i} q^{ix}}{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^k (k-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k)},$$

where  $n, k$  are positive integers. From (27) we note that

$$(28) \quad \beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{\binom{i+k}{k}_q [k]_q!}{\binom{i+k}{k}} \frac{[k]_q!}{k!} q^{ix}.$$

It is easy to see that

$$(29) \quad \frac{\binom{k}{j}}{\binom{j+n}{n} n!} = \frac{(k+n) \cdots (k+1) k \cdots (k-j+1)}{(j+n)! (k+n) \cdots (k+1)} = \frac{\binom{k+n}{k-j}}{\binom{k+n}{n} n!}.$$

By (27), (28) and (29) we obtain the following theorem.

**Theorem 5.** For  $n, k \in \mathbb{Z}_+$ , we have

$$(30) \quad \beta_{n,q}^{(-k)}(x) = \frac{1}{(1-q)^n} \sum_{i=0}^n (-1)^i \binom{i+k}{k} \frac{\binom{n+k}{n-i}_q [k]_q!}{\binom{n+k}{k}} \frac{[k]_q!}{k!} q^{ix}.$$

From (18) and (30) we derive

$$s_2(n, k, q) = \binom{k+n}{n} \frac{n!}{[n]_q!} \beta_{k,q}^{(-k)}(0).$$

That is,

$$\frac{1}{(1-q)^k} \sum_{i=0}^k \frac{(-1)^k \binom{k}{i}}{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{l=1}^n (n-l+i)x_l} d\mu_q(x_1) \cdots d\mu_q(x_n)} = \frac{[n]_q!}{\binom{k+n}{n} n!} s_2(n, k, q).$$

Thus, we note that

$$(31) \quad \beta_{0,q}^{(-k)}(0) = \left( \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^k (k-i)x_i} d\mu_q(x_1) \cdots d\mu_q(x_k) \right)^{-1} = \frac{[k]_q!}{k!}.$$

By the same method we see that  $\beta_{1,q}^{(2)}(0) = \frac{-2(q+2)}{[2]_q [3]_q}, \dots, \beta_{0,q}^{(-k)}(0) = \frac{[k]_q!}{k!}$ . Thus, we have

$$s_2(k, 0, q) = \frac{k!}{[k]_q!} \beta_{0,q}^{(-k)}(0) = \frac{k!}{[k]_q!} \frac{[k]_q!}{k!} = 1.$$

From the definition of  $\beta_{m,q}^{(k)}$  we can also derive the following equality.

$$\begin{aligned} & \sum_{i=0}^m \binom{m}{i} (q-1)^i \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^i q^{\sum_{l=1}^k (k-l)x_l} d\mu_q(x_1) \cdots d\mu_q(x_k) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{(m+k-1)x_1 + \cdots + (m+1)x_{k-1} + mx_k} d\mu_q(x_1) \cdots d\mu_q(x_k) = \frac{\binom{m+k}{k}_q}{\binom{m+k}{k}_q} \frac{k!}{[k]_q!}. \end{aligned}$$

Therefore we obtain the following:

$$\sum_{i=0}^m \binom{m}{i} (q-1)^i \beta_{i,q}^{(k)} = \frac{\binom{m+k}{k}_q}{\binom{m+k}{k}_q} \frac{k!}{[k]_q!}.$$

Finally, we observe that

$$q^{\binom{n}{2}} [x]_{n,q} = [x]_q q [x-1]_q \cdots q^{n-1} [x-n+1]_q = [x]_q \cdot ([x]_q - 1) \cdots ([x]_q - [n-1]_q).$$

Thus, we have

$$q^{\binom{n}{2}} \binom{x}{n}_q = \frac{1}{[n]_q!} \prod_{k=0}^n ([x]_q - [k]_q) = \frac{1}{[n]_q!} \sum_{k=0}^n s_1(n, k, q) [x]_q^k.$$

#### §4. Further Remarks and Observations

In this section, let  $p$  be a fixed odd prime number. For  $n \in \mathbb{N}$ ,  $k \in \mathbb{Z}_+$ , we consider the following  $q$ -Euler numbers of higher order.

$$(32) \quad E_k^{(n)}(x, q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ \sum_{i=1}^n x_i + x \right]_q^k q^{\sum_{j=1}^n x_j (n-j)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_n),$$

where  $\mu_{-q}(x + p^N \mathbb{Z}_p) = \frac{1+q}{1+q^{p^N}} (-q)^x = \frac{(-q)^x}{[p^N]_{-q}}$ , see [10]. From (32) we note that

$$E_n^{(k)}(x, q) = \frac{[2]_q^n}{(1-q)^k} \sum_{l=0}^k \binom{k}{l} \frac{(-1)^l q^{lx}}{(1+q^{n+l}) \cdots (1+q^{l+1})}.$$

The  $q$ -binomial formulae are known as

$$(33) \quad \prod_{i=1}^n (a + bq^{i-1}) = \sum_{k=0}^n \binom{n}{k}_q q^{\binom{k}{2}} q^{n-k} b^k,$$

and

$$\prod_{i=1}^n (a - bq^{i-1})^{-1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q b^k.$$

By (32) and (33) we obtain the following:



**Proposition 6.** For  $n \in \mathbb{N}$ ,  $k \in \mathbb{Z}_+$ , we have

$$E_k^{(n)}(x, q) = \frac{[2]_q^n}{(1-q)^k} \sum_{l=0}^k \binom{k}{l} (-1)^l q^{lx} \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q (-1)^i q^{(l+1)i}.$$

Seeking to define a suitable polynomial analogue for negative value of  $n$ , we give the definition as follows:

$$(34) \quad E_k^{(-n)}(x, q) = \frac{1}{(1-q)^k} \frac{1}{[2]_q^n} \sum_{l=0}^k \binom{k}{l} (-1)^l q^{lx} \left( \prod_{i=1}^n (1 + q^{l+i}) \right).$$

From (33) and (34) we can also derive the following Eq.(35).

$$(35) \quad E_k^{(-n)}(x, q) = \frac{1}{(1-q)^k} \frac{1}{[2]_q^n} \sum_{l=0}^k \binom{k}{l} (-1)^l q^{lx} \sum_{i=0}^n \binom{n}{i}_q q^{\binom{i}{2}} q^{(l+1)i}.$$

#### REFERENCES

- [1] I. N. Cangul, V. Kurt, Y. Simsek, H.K. Pak, S.-H. Rim, *An invariant  $p$ -adic  $q$ -integral associated with  $q$ -Euler numbers and polynomials*, J. Nonlinear Math. Phys. **14** (2007), 8–14.
- [2] L. C. Carlitz,  *$q$ -Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), 987–1000.
- [3] L. C. Carlitz, *Expansions of  $q$ -Bernoulli numbers*, Duke Math. J. **25** (1958), 355–364.
- [4] M. Cenkci, M. Can and V. Kurt,  *$p$ -adic interpolation functions and Kummer-type congruences for  $q$ -twisted Euler numbers*, Advan. Stud. Contemp. Math. **9** (2004), 203–216.
- [5] M. Cenkci, M. Can, *Some results on  $q$ -analogue of the Lerch zeta function*, Adv. Stud. Contemp. Math. **12** (2006), 213–223.
- [6] A.S. Hegazi, M. Mansour, *A note on  $q$ -Bernoulli numbers and polynomials 13 (2006)*, 9–18, J. Nonlinear Math. Phys. **13** (2006), 9–18.
- [7] T. Kim, *On  $p$ -adic  $q$ - $l$ -functions and sums of powers*, J. Math. Anal. Appl. **329** (2007), 1472–1481.
- [8] T. Kim,  *$q$ -Volkenborn integration*, Russ. J. Math. Phys. **9** (2002), 288–299.
- [9] T. Kim, *A Note on  $p$ -Adic  $q$ -integral on  $\mathbb{Z}_p$  Associated with  $q$ -Euler Numbers*, Adv. Stud. Contemp. Math. **15** (2007), 133–138.
- [10] T. Kim, *On  $p$ -adic interpolating function for  $q$ -Euler numbers and its derivatives*, J. Math. Anal. Appl. **339** (2008), 598–608.
- [11] T. Kim,  *$q$ -Extension of the Euler formula and trigonometric functions*, Russ. J. Math. Phys. **14** (2007), 275–278.
- [12] T. Kim, *Power series and asymptotic series associated with the  $q$ -analog of the two-variable  $p$ -adic  $L$ -function*, Russ. J. Math. Phys. **12** (2005), 186–196.
- [13] T. Kim, *Non-Archimedean  $q$ -integrals associated with multiple Changhee  $q$ -Bernoulli polynomials*, Russ. J. Math. Phys. **10** (2003), 91–98.
- [14] B. A. Kupersmidt, *Reflection symmetries of  $q$ -Bernoulli polynomials*, J. Nonlinear Math. Phys. **12** (2005), 412–422.
- [15] H. Ozden, Y. Simsek, S.-H. Rim, I.N. Cangul, *A note on  $p$ -adic  $q$ -Euler measure*, Adv. Stud. Contemp. Math. **14** (2007), 233–239.
- [16] C.S. Ryoo, *A note on  $q$ -Bernoulli numbers and polynomials*, Appl. Math. Lett. **20** (2007), 524–531.

- [17] C.S. Ryoo, *The zeros of the generalized twisted Bernoulli polynomials*, Adv. Theor. Appl. Math. **1** (2006), 143–148.
- [18] J. Satoh,  *$q$ -analogue of Riemann's  $\zeta$ -function and  $q$ -Euler numbers*, J. Number Theory **31** (1989), 346–362.
- [19] M. Schork, *Ward's "calculus of sequences",  $q$ -calculus and the limit  $q \rightarrow -1$* , Adv. Stud. Contemp. Math. **13** (2006), 131–141.
- [20] M. Schork, *Combinatorial aspects of normal ordering and its connection to  $q$ -calculus*, Adv. Stud. Contemp. Math. **15** (2007), 49–57.
- [21] Y. Simsek, *On twisted  $q$ -Hurwitz zeta function and  $q$ -two-variable  $L$ -function*, Appl. Math. Comput. **187** (2007), 466–473.
- [22] Y. Simsek, *On  $p$ -adic twisted  $q$ - $L$ -functions related to generalized twisted Bernoulli numbers*, Russ. J. Math. Phys. **13** (2006), 340–348.
- [23] Y. Simsek, *Twisted  $(h, q)$ -Bernoulli numbers and polynomials related to twisted  $(h, q)$ -zeta function and  $L$ -function*, J. Math. Anal. Appl. **324** (2006), 790–804.
- [24] Y. Simsek, *Theorems on twisted  $L$ -function and twisted Bernoulli numbers*, Advan. Stud. Contemp. Math. **11** (2005), 205–218.
- [25] H. M. Srivastava, T. Kim and Y. Simsek,  *$q$ -Bernoulli numbers and polynomials associated with multiple  $q$ -zeta functions and basic  $L$ -series*, Russ. J. Math. Phys. **12** (2005), 241–268.

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